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1979 J. Phys. A: Math. Gen. 12 1771

(http://iopscience.iop.org/0305-4470/12/10/023)

J. Phys. A: Math. Gen., Vol. 12, No. 10, 1979. Printed in Great Britain

Equations of motion in linearised gravity: III Rotating sources

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Received 13 February 1979

Abstract. We consider a line-element which in linearised gravity we associate with the gravitational field of an arbitrarily accelerating, rotating, axially symmetric source. We establish that if the body rotates about its symmetry axis and moves along its symmetry axis when viewed in the background flat space–time and if its linearised field (Riemann tensor) is free of wire singularities then the body must move with uniform acceleration and rotation and its mass must be constant.

1. Introduction

In a recent series of papers, Hogan and Imaeda (1979a, b, c) have adopted a new approach to study the motion of point sources of some algebraically special linearised vacuum gravitational and Einstein–Maxwell fields with twist-free diverging degenerate principal null direction (cf Robinson and Trautman (1962)). Their method consists in expanding the Robinson–Trautman line-element about Minkowskian space-time in powers of the mass parameter (assumed small) and obtaining the field equations to be satisfied by the perturbation of the metric from flat space–time. Specifying the source world–line in the background Minkowskian space–time, these equations are integrated, functions of integration being determined by the requirement that terms be excluded from the linearised field (Riemann tensor) of the particle which are singular along null-rays emanating into the future from events on the source world-line in the background space–time. In the case of an unchanged source, they established that if the motion of the particle is rectilinear in an inertial frame of reference and if the field is directional singularity-free, then the particle must move with uniform acceleration.

Robinson and Robinson (1969) have extended the Robinson and Trautman (1962) work to the case in which the degenerate principal null direction is twisting. It is compelling, by analogy with the twist-free case, to think of the Robinson and Robinson (1969) fields as containing the fields of arbitrarily moving, rotating bodies. In an earlier paper, using the method developed in Hogan and Imaeda (1979a, b, c) Hogan and O'Brien (1979) examined a member of the Robinson and Robinson (1969) family of line-elements with this interpretation in mind. In Hogan and O'Brien (1979), the source considered is axially symmetric, rotating uniformly and slowly about its symmetry axis, moving with arbitrary acceleration along its symmetry axis and its mass is small and constant. The purpose of the present paper is to consider in more detail the line-element associated with a rotating, accelerating source. The results described in Hogan and O'Brien (1979) are generalised by allowing the mass and rotation to vary.

0305-4470/79/101771+09$01.00

In § 2 we introduce the exact Robinson and Robinson (1969) family of solutions. In § 3 the background Minkowskian space–time is briefly discussed and the linearisation procedure explained. A line-element is obtained which is the line-element of Minkowskian space–time plus a small first-order perturbation. In § 4 we specify the background source world-line and assume that the source is axially symmetric, rotating (in a technical sense) slowly about its symmetry axis and moving along its symmetry axis when viewed in the background space–time. This enables us to solve the linearised field equations. We determine functions of integration by the requirement that the field of the body be wire singularity-free. This also establishes that the mass of the source must be constant and its acceleration and rotation uniform. In § 5 we present the final form of the line-element.

2. The Robinson and Robinson line-element

Robinson and Robinson (1969) have shown that if a space–time admits a null vector field k^i (i = 1, 2, 3, 4) tangent to a shear-free diverging congruence of affinely parametrised null geodesic curves then coordinates $x^i = (\zeta, \bar{\zeta}, \sigma, \rho)$ may be chosen such that in a vacuum the 'main' gravitational field equations (in the Sachs (1962) classification) have the solution

$$ds^2 = 2P\bar{P}\,d\zeta\,d\bar{\zeta} + 2\,d\Sigma(d\rho + Z\,d\zeta + \bar{Z}\,d\bar{\zeta} + S\,d\Sigma), \tag{2.1a}$$

$$d\Sigma = -(d\sigma + i\,\partial B\,d\zeta - i\bar{\partial}B\,d\bar{\zeta}) = k_i dx^i, \tag{2.1b}$$

$$P = \exp(u)(\rho + i\Omega), \tag{2.1c}$$

$$S = -\rho\dot{u} - \tfrac{1}{2}K + (\rho m + \Omega M)/(\rho^2 + \Omega^2), \tag{2.1d}$$

$$Z = \rho\Lambda - i(D + \Lambda)\Omega, \tag{2.1e}$$

$$\Omega = -\tfrac{1}{2}\exp(-2u)\{D\,\bar{\partial} + \bar{D}\,\partial\}B, \tag{2.1f}$$

$$\Lambda = -i\,\partial\dot{B}, \tag{2.1g}$$

$$K = \exp(-2u)(\bar{D}L + D\bar{L}), \tag{2.1h}$$

$$L = \Lambda - Du, \tag{2.1i}$$

$$M = K\Omega + \tfrac{1}{2}\exp(-2u)\{(\bar{D} + \bar{\Lambda})(D + \Lambda) + (D + \Lambda)(\bar{D} + \bar{\Lambda})\}\Omega, \tag{2.1j}$$

where B, u and m are functions of ζ, $\bar{\zeta}$, σ only. A bar denotes complex conjugation and, for any function $f(\zeta, \bar{\zeta}, \sigma)$ we use the notation

$$Df = \partial f - i\,\partial B\dot{f}, \qquad \bar{D}f = \bar{\partial}f + i\,\bar{\partial}B\dot{f}, \tag{2.2}$$

where $\partial f = \partial f/\partial\zeta$ and $\bar{\partial}f = \partial f/\partial\bar{\zeta}$ and a dot indicates partial differentiation with respect to σ.

The remaining 'subsidiary' field equation can be written as

$$A = 0, \qquad \bar{A} = 0, \tag{2.3}$$

$$\exp(-u)I - \{\exp(3u)(m + iM)\}^{\cdot} = 0, \tag{2.4}$$

where we define

$$A = (D + 3\Lambda)(m - iM), \tag{2.5}$$

$$I = (\bar{D}^2 + 2\bar{L}\bar{D})J, \tag{2.6}$$

$$J = DL + L^2. \tag{2.7}$$

3. Linearisation

It can be shown (see, for example, Hogan and Imaeda (1979a)) that when the Minkowskian line–element is written in Robinson–Trautman form (cf the line–element (2.1) with $B = 0$), that it is given explicitly by

$$\mathrm{d}s^2 = 2\rho^2 \exp(2\underset{0}{u})\, \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta} - 2\, \mathrm{d}\rho\, \mathrm{d}\sigma - (1 + 2\underset{0}{\dot{u}}\rho)\, \mathrm{d}\sigma^2, \tag{3.1}$$

where

$$\exp(-\underset{0}{u}) = \lambda^4(1 + \tfrac{1}{2}\zeta\bar{\zeta}) - \lambda^3(1 - \tfrac{1}{2}\zeta\bar{\zeta}) - (\lambda^1 - \mathrm{i}\lambda^2)2^{-1/2}\zeta - (\lambda' + \mathrm{i}\lambda^2)2^{-1/2}\zeta, \tag{3.2a}$$

$$\underset{0}{\dot{u}} = \mu^i k_i. \tag{3.2b}$$

The λ^i are functions of σ only and satisfy $(\lambda')^2 + (\lambda^2)^2 + (\lambda^3)^2 - (\lambda^4)^2 = -1$. They have the following interpretation: if X^i are rectangular Cartesian coordinates and time in this background space–time, then $\rho = 0$ is a time-like world–line with equation $X^i = x^i(\sigma)$. In these coordinates $\lambda^i = \mathrm{d}x^i/\mathrm{d}\sigma$ are the components of its 4-velocity and its 4-acceleration is $\mu^i = \mathrm{d}\lambda^i/\mathrm{d}\sigma$. k^i is tangent to the future-pointing generators of the null cones at every event on $\rho = 0$ (they will have equations $\sigma =$ constant) and is normalised, so that $k^i\lambda_i = -1$.

We now proceed to expand the Robinson and Robinson line-element (2.1) about the background solution (3.1). The function m in (2.1) is interpreted as the mass of the linearised source. We shall assume that m is small of first order, writing $m = 0_1$ and expand u and B in the form

$$u = \underset{0}{u} + \underset{1}{u}, \tag{3.3a}$$

$$B = \underset{1}{B} + \underset{2}{B}, \tag{3.3b}$$

where a subscript n ($n = 0, 1, 2, \ldots$) under any quantity means that it is small of nth order. Substituting these into (2.1f), (2.1g), (2.1i) and (2.1i) yields

$$K = \underset{0}{K} + \underset{1}{K}, \tag{3.4a}$$

$$\Omega = \underset{1}{\Omega} + \underset{2}{\Omega}, \tag{3.4b}$$

with

$$\underset{0}{K} = -\Delta\underset{0}{u}, \tag{3.5a}$$

$$\underset{1}{\Omega} = -\tfrac{1}{2}\Delta\underset{1}{B}, \tag{3.5b}$$

where $\Delta = 2\exp(-2\underset{0}{u})\partial\bar{\partial}$. One can easily show that with $\underset{0}{u}$ given by (3.2a), $\Delta\underset{0}{u} = -1$ and so $\underset{0}{K}$ given by (3.5a) is unity. Now, substitution of the above expressions into (2.1j) yields

$$M = \tfrac{1}{2}\Delta(\underset{1}{\Omega} - \underset{1}{B}) + O_2. \tag{3.6}$$

4. A rotating axially symmetric source

We think of the source in the background space–time as a time-like world–tube filled with time-like world–lines parallel to $\rho = 0$ (for a detailed description of the Kerr solution from this point of view see Hogan 1975). We shall assume that the source is axially symmetric with the X^3 axis as symmetry axis and that it moves along this axis, i.e. $\lambda^1 = \lambda^2 = 0$. Then (3.2) becomes

$$\exp(-\underset{0}{u}) = \exp(\alpha)(\tfrac{1}{2}\zeta\bar{\zeta} + \exp(-2\alpha)), \tag{4.1a}$$

$$\underset{0}{\dot{u}} = -\dot{\alpha}\frac{(\tfrac{1}{2}\zeta\bar{\zeta} - \exp(-2\alpha))}{(\tfrac{1}{2}\zeta\bar{\zeta} + \exp(-2\alpha))}, \tag{4.1b}$$

where $\alpha = \alpha(\sigma)$ is defined by $\exp(\alpha) = \lambda^3 + \lambda^4$ (thus $\lambda^3 - \lambda^4 = -\exp(-\alpha)$) so that $\dot{\alpha} = (\mu^i \mu_i)^{1/2}$ is the magnitude of the 4-acceleration of the source.

We guarantee the axial symmetry of the source by requiring that the functions m, u and B, which appear in the metric (2.1), depend on ζ and $\bar{\zeta}$ in the combination $\zeta\bar{\zeta}$ as in (4.1). one can then show that the vector

$$i(\zeta\partial - \bar{\zeta}\bar{\partial}) \tag{4.2}$$

satisfies Killing's equations. Introducing polar coordinates θ, ϕ via $\zeta = 2^{1/2}\exp(i\phi)\tan(\theta/2)$, (4.2) becomes $\partial/\partial\phi$, thus verifying that line–elements of the form (2.1), satisfying this requirement, are axially symmetric. In what follows it is convenient to introduce the new variable

$$\xi = \frac{\tfrac{1}{2}\zeta\bar{\zeta} - \exp(-2\alpha)}{\tfrac{1}{2}\zeta\bar{\zeta} + \exp(-2\alpha)}. \tag{4.3}$$

With a view to the eventual recovery of the linearised Kerr solution, as a special case we choose

$$\underset{1}{\Omega} = \underset{1}{B} \tag{4.4}$$

Substitution of (4.4) into (3.5b) and (3.6) yields

$$\Delta\underset{1}{\Omega} + 2\underset{1}{\Omega} = 0, \tag{4.5a}$$

$$M = O_2. \tag{4.5b}$$

The first of these has the solution

$$\underset{1}{\Omega} = c(\sigma)P_1(\xi) + \mathrm{d}(\sigma)Q_1(\xi), \tag{4.6}$$

where c and d are functions of integration and $P_1(\xi)$ and $Q_1(\xi)$ are $l=1$ Legendre functions of the first and second kind. At the outset we require that Ω be singularity-free. Since $Q_1(\xi)$ has singularities at $\xi=\mp 1$ (the geometrical interpretation of these as 'wire' or 'directional' singularities is described following (4.15) below) we must take $d=0$ in (4.6). Thus we have

$$\underset{1}{\Omega}=c(\sigma)\xi \tag{4.7}$$

with c small of first order. We note that the twist of the vector field in (2.1b) is given by $\omega=\rho^{-2}(\underset{1}{\Omega}+\underset{2}{\Omega})+O_3$. Thus we are here ensuring that this quantity is free from singularities at $\xi=\pm 1$.

Using (4.5*b*), we find that the 'subsidiary' condition (2.3) is satisfied provided

$$\partial m/\partial\xi=c\dot{M}+3(\dot{c}-2c\dot{\alpha}\xi)M+O_4, \tag{4.8a}$$

$$\partial M/\partial\xi=-c\dot{m}-3m\dot{c}+6mc\dot{\alpha}\xi+O_3. \tag{4.8b}$$

It follows from the first of these that

$$m=\underset{1}{m}(\sigma)+O_3. \tag{4.9}$$

In the special case of $\alpha=0$ and c and m both having constant values (and thus $\underset{1}{u}=0=\underset{1}{B}$) the line–element (2.1) becomes the Kerr solution with mass m and angular momentum mc. In general we interpret (2.1) (with the above restrictions) as the line–element associated with a slowly rotating, arbitrarily accelerating, axially symmetric source, rotating non-uniformly about and moving along its symmetry axis when viewed in the flat background space–time.

Substitution of (4.1), (4.4) and (4.7) into (2.1*k*) yields

$$\underset{1}{K}=(\Delta\underset{1}{u}+2\underset{1}{u})+O_2. \tag{4.10}$$

Using this and substituting into the subsidiary condition (2.4) we find

$$(1/4)\Delta\underset{1}{K}-\underset{1}{\dot{m}}-3\underset{1}{m}\underset{0}{\dot{u}}=O_2. \tag{4.11}$$

The tetrad components of the linearised Riemann tensor are, in Newman–Penrose (1966) notation,

$$\begin{aligned}
&\Psi_0=\Psi_1=0,\\
&\Psi_2=\frac{-\underset{1}{m}}{(\rho+\mathrm{i}\Omega)^3}+O_2,\\
&\Psi_3=\frac{\bar{\zeta}\exp(\underset{0}{u})}{2(\rho+\mathrm{i}\Omega)^2}\frac{\partial\underset{1}{K}}{\partial\xi}+O_2,\\
&\Psi_4=\frac{\bar{\zeta}^2\exp(2u)}{(\rho+\mathrm{i}\Omega)}\frac{\partial}{\partial\xi}\left\{\frac{\partial\underset{1}{\dot{u}}}{\partial\xi}+2\dot{\alpha}\underset{1}{u}\right\}+\frac{\bar{\zeta}^2\exp(-2\underset{0}{u})}{2(\rho+\mathrm{i}\Omega)^2}\frac{\partial^2\underset{1}{K}}{\partial\xi^2}+O_2.
\end{aligned} \tag{4.12}$$

In arriving at (4.10)–(4.12), some remarkable cancellations have occured so that the only place in which the rotation parameter c appears is in inverse powers of $(\rho+\mathrm{i}\Omega)$ in (4.12). Otherwise the problem of determining solutions of (4.10) and (4.11) free of wire singularities in (4.12) is exactly the same problem one faces in the twist-free case.

Substituting (4.1) and (4.3), the equation (4.11) becomes in terms of the variable ξ.

$$\frac{\partial}{\partial\xi}\left\{(1-\xi^2)\frac{\partial \underset{1}{K}}{\partial\xi}\right\}=4\underset{1}{\dot{m}}-12\,\underset{1}{m}\dot{\alpha}\xi+O_2. \tag{4.13}$$

This can be integrated easily to give

$$\underset{1}{K}=-2\underset{1}{\dot{m}}\ln(1-\xi^2)+6\underset{1}{m}\dot{\alpha}\xi+\tfrac{1}{2}N(\sigma)\ln[1+\xi/1-\xi]+R(\sigma)+O_2, \tag{4.14}$$

where R and N are functions of integration. On substituting this into Ψ_3 in (4.12), we find

$$\Psi_3=\frac{\bar{\zeta}\exp(\underset{0}{u})}{2(\rho+\mathrm{i}\Omega)^2}\left\{\frac{4\underset{1}{\dot{m}}\xi}{1-\xi^2}+6\underset{1}{m}\dot{\alpha}+\frac{N(\sigma)}{1-\xi^2}\right\}+O_2. \tag{4.15}$$

This is singular at $\rho=0$, $\Omega=0$—the analogue of the "Kerr circle". It is also singular at $\xi=\pm1$. By (3.2*b*), (4.1*b*) and (4.3), $\xi=\pm1$ corresponds to

$$\mu^i k_i=\mp\dot{\alpha}, \tag{4.16}$$

so that, in the background space–time, for each constant value of σ, (4.16) picks out a pair of diametrically opposed future-pointing null-rays k^i on every future-pointing null cone at each event on the background source world-line. Along these rays (4.15) is singular. $\xi=-1$ corresponds to $\zeta\bar{\zeta}=0$ or $\theta=0$ while $\xi=+1$ corresponds to $\zeta\bar{\zeta}\to\infty$ or $\theta=\pi$, for each fixed value of σ. Choosing $N=4\underset{1}{\dot{m}}$, the singularity at $\xi=-1$ is removed while Ψ_3 remains singular at $\xi=+1$. Similarly the choice of $N=-4\underset{1}{\dot{m}}$ removes the singularity at $\xi=+1$ but not at $\xi=-1$. To exclude both singularities we must take $N=0$ and

$$\underset{1}{\dot{m}}=0, \tag{4.17}$$

so that $\underset{1}{K}$ is given by

$$\underset{1}{K}=6\underset{1}{m}\dot{\alpha}\xi+R(\sigma)+O_2, \tag{4.18}$$

with $\underset{1}{m}$ constant.

We now turn to (4.10). in terms of the variable ξ this equation becomes

$$\frac{\partial}{\partial\xi}\left\{(1-\xi^2)\frac{\partial \underset{1}{u}}{\partial\xi}\right\}+2\underset{1}{u}=-6\underset{1}{m}\dot{\alpha}\xi-R(\sigma)+O_2, \tag{4.19}$$

Integrating this, we obtain

$$\underset{1}{u}=\underset{1}{m}\dot{\alpha}\xi\ln(1-\xi^2)-\tfrac{1}{2}R(\sigma)+S(\sigma,\xi)+O_2, \tag{4.20a}$$

$$S(\sigma,\xi)=\epsilon(\sigma)P_1(\xi)+\delta(\sigma)Q_1(\xi), \tag{4.20b}$$

where ϵ and δ are functions of integration and $P_1(\xi)$ and $Q_1(\xi)$ are $l=1$ Legendre functions of the first and second kind.

Substituting (4.18) and (4.20) into Ψ_4 given by (4.12), we find

$$\Psi_4 = \frac{\bar{\zeta}^2 \exp(2\underset{0}{u})}{(\rho+\mathrm{i}\Omega)}\left\{-6\underset{1}{m}\dot{\alpha}^2 - 2\underset{1}{m}\ddot{\alpha}\xi\frac{(3-\xi^2)}{(1-\xi^2)^2} + \frac{2\dot{\delta}}{(1-\xi^2)^2}\right\} + O_2. \tag{4.21}$$

Ψ_4 is singular on $\rho=0$, $\Omega=0$ and also on $\xi=\pm 1$ unless $\dot{\delta}=0$ and $\ddot{\alpha}=0$. To exclude these 'directional' singularities we choose $\delta=$ constant and $\dot{\delta}=$ constant $=a$. Hence the tetrad components of the linearised Riemann tensor are finally given by

$$\Psi_0 = \Psi_1 = 0, \qquad \Psi_2 = \frac{-\underset{1}{m}}{(\rho+\mathrm{i}\Omega)^3} + O_2,$$

$$\Psi_3 = \frac{\bar{\zeta}\exp(\underset{1}{u})}{(\rho+\mathrm{i}\Omega)_2}(3\underset{1}{m}a) + O_2, \qquad \Psi_4 = \frac{-\bar{\zeta}^2\exp(2\underset{0}{u})}{(\rho+\mathrm{i}\Omega)}(6\underset{1}{m}a^2) + O_2.$$

These are only singular on $\rho=0$, $\Omega=0$. In addition

$$2\Psi_3^2 - 3\Psi_2\Psi_4 = O_3, \tag{4.23}$$

so that the linearised field is type D in the Petrov classification.

The functions of integration $R(\sigma)$, $\epsilon(\sigma)$, δ do not appear in (4.22). As Hogan and Imaeda (1979a, b) pointed out, they can be removed by a gauge transformation

$$\underset{1}{K} \to \underset{1}{K} - R(\sigma), \qquad \underset{1}{u} \to \underset{1}{u} + \tfrac{1}{2}R(\sigma) - \epsilon(\sigma)P_1(\xi) - \delta Q_1(\xi). \tag{4.24}$$

Applying (4.24) to (4.18) and (4.20) it is clear that we can put

$$\begin{aligned} \underset{1}{K} &= 6\underset{1}{m}a\xi + O_2, \\ \underset{1}{u} &= \underset{1}{m}a\xi \ln(1-\xi^2) = O_2 \end{aligned} \tag{4.25}$$

where $a=\dot{\alpha}$ is the (uniform) acceleration of the source.

We now consider in more detail the subsidiary conditions (2.3) which are given by equation (4.8) for the case we are considering. Using (4.19) and substituting $\dot{\alpha}=a$, (4.8*b*) can be integrated to give

$$M = -3\underset{1}{m}\dot{c}\xi + 3\underset{1}{m}ca\xi^2 + \beta(\sigma) + O_3. \tag{4.26}$$

where $\beta = O_2$ is a function of integration. Also, by (2.1*j*) we have

$$M = \{\underset{1}{K} + 2\underset{1}{u}\}c\xi + \underset{2}{\Omega} + \tfrac{1}{2}\Delta\underset{2}{\Omega} + O_3. \tag{4.27}$$

Combining (4.26) with (4.27) and substituting from (4.25), we obtain the following equation for $\underset{2}{\Omega}$

$$\frac{\partial}{\partial\xi}\left\{(1-\xi^2)\frac{\partial\underset{2}{\Omega}}{\partial\xi}\right\} + 2\underset{2}{\Omega} = -6\underset{1}{m}\{ca\xi^2 + \dot{c}\xi\} - 4\underset{1}{m}ca\xi^2\ln(1-\xi^2) + 2\beta(\sigma) + O_3. \tag{4.28}$$

This can be integrated to give

$$\underset{2}{\Omega} = \{\tfrac{3}{2}\underset{1}{m}ca - \gamma(\sigma)\}\left\{1 - \tfrac{1}{2}\xi \ln\left[\frac{1+\xi}{1-\xi}\right]\right\} - \underset{1}{m}ca\xi^2$$

$$+ \underset{1}{m}\dot{c}\xi \ln(1-\xi^2) - \underset{1}{m}ca(1-\xi^2)\ln(1-\xi^2) + \nu(\sigma)\xi + \beta(\sigma) + O_3, \qquad (4.29)$$

where the functions of integration ν and γ are small of second order and appear in (4.29) as coefficients of $l = 1$ Legendre functions of the first and second kind.

The first and third terms in (4.29) are singular at $\xi = \pm 1$. We can rewrite (4.29) in the form

$$\underset{2}{\Omega} = -\underset{1}{m}ca\xi^2 - \underset{1}{m}ca(1-\xi^2)\ln(1-\xi^2) + \nu(\sigma)\xi + \tau(\sigma)$$

$$+ \xi(\underset{1}{m}\dot{c} - \tfrac{1}{2}\{\tfrac{3}{2}\underset{1}{m}ca - \gamma\})\ln(1+\xi)$$

$$+ \xi(\underset{1}{m}\dot{c} + \tfrac{1}{2}\{\tfrac{3}{2}\underset{1}{m}ca - \gamma\})\ln(1-\xi) + O_3, \qquad (4.30)$$

where $\tau(\sigma) = \beta(\sigma) + \tfrac{3}{2}\underset{1}{m}ca - \gamma$. It is clear that the final pair of terms in (4.30) are singular at either $\xi = -1$ or $+1$. But initially, in seeking a solution $\underset{1}{\Omega}$ of (4.5*a*), we required that Ω be singularity-free. Choosing $\gamma = \tfrac{3}{2}\underset{1}{m}ca - 2\underset{1}{m}\dot{c}$ we remove the first singular term in (4.30) but $\underset{2}{\Omega}$ is still singular at $\xi = +1$. Similarly the choice of $\gamma = \tfrac{3}{2}\underset{1}{m}ca + 2\underset{1}{m}\dot{c}$ removes the singularity in $\underset{2}{\Omega}$ at $\xi = +1$ but not at $\xi = -1$. Clearly, in order to remove both singularities we must take $\gamma = \tfrac{3}{2}\underset{1}{m}ca$ and

$$\dot{c} = 0. \qquad (4.31)$$

Thus at first order the source is rotating uniformly.

Having determined $\underset{1}{K}$, $\underset{1}{u}$, and $\underset{2}{\Omega}$, we could now evaluate the second order correction to B from (2.1*f*) and, using (4.8*a*), the corrections to m up to and including third order terms. However, to obtain the complete second metric perturbation, the 'subsidiary' field equation (2.4) would have to be solved for $\underset{2}{K}$ and $\underset{2}{u}$.

5. Discussion

Starting with a line-element contained in the Robinson and Robinson (1969) family, which is interpreted as describing the exterior field of a rotating, accelerating, axially symmetric source, we have shown that if the linearised field (Riemann tensor) of the body and the linearised twist of the degenerate principal null direction of the Riemann tensor are free from wire singularities, then the body must move with uniform acceleration and, to first order, its mass must be constant and its rotation uniform.

The final form of the line-element in the linear approximation is given by (2.1a) with

$$\mathrm{d}\Sigma = -\frac{\mathrm{i}c(1-\xi^2)}{2\zeta}\mathrm{d}\zeta + \frac{\mathrm{i}c(1-\xi^2)}{2\bar{\zeta}}\mathrm{d}\bar{\zeta} - \mathrm{d}\sigma,$$

$$P = \tfrac{1}{2}\exp(a\sigma)(1-\xi)\{1+\underset{1}{m}a\xi\ln(1-\xi^2)\}(\rho+\mathrm{i}c\xi)+O_2,$$

$$Z = -\frac{\mathrm{i}c(1-\xi^2)}{2\zeta}(1-2\rho\xi a)+O_2, \tag{5.1}$$

$$S = \rho\{a\xi+\underset{1}{m}a^2[2\xi^2-(1-\xi^2)\ln(1-\xi^2)]\}-\tfrac{1}{2}(1+6\underset{1}{m}a\xi)+\underset{1}{m}\rho/(\rho^2+c^2\xi^2)+O_2,$$

where ξ is given by (4.3). The final form of the linearised Riemann tensor components is given by (4.22), with $\underset{1}{m}$ and $c=\xi^{-1}\underset{1}{\Omega}$ both constant.

As $\sigma\to\pm\infty$, $\exp(\underset{0}{u})\to 0$ for both positive and negative values of a. Hence in the infinite past or infinite future the only non-vanishing (modulo an O_2 error) tetrad component of the Riemann tensor is Ψ_2. When viewed in the background Minkowskian space–time, the rotating source appears to be travelling with the speed of light (since the 4 velocity components $\lambda^i(\sigma)$ become infinite in this limit).

If $a=0$, (2.1a) with (5.1) becomes a form of the Kerr solution given by Hogan (1977). In these coordinates the exact Kerr solution also satisfies the linearised vacuum field equations. if in addition we put $c=0$, we recover the Schwarzschild solution.

Acknowledgments

I am indebted to Dr P A Hogan for his interest and encouragement and for many helpful suggestions.

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